

# Maximal $L^2$ regularity for Ornstein-Uhlenbeck equation in convex sets of Banach spaces

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## Abstract

We study the elliptic equation  $\lambda u - L^\Omega u = f$  in an open convex subset  $\Omega$  of an infinite dimensional separable Banach space  $X$  endowed with a centered non-degenerate Gaussian measure  $\gamma$ , where  $L^\Omega$  is the Ornstein-Uhlenbeck operator. We prove that for  $\lambda > 0$  and  $f \in L^2(\Omega, \gamma)$  the weak solution  $u$  belongs to the Sobolev space  $W^{2,2}(\Omega, \gamma)$ . Moreover we prove that  $u$  satisfies the Neumann boundary condition in the sense of traces at the boundary of  $\Omega$ . This is done by finite dimensional approximation.

## 1 Introduction

Let  $X$  be a separable Banach space, let  $\gamma$  be a centered non-degenerate Gaussian measure in  $X$  with covariance  $Q$ , and let  $H = Q^{1/2}(X)$  be the associated Cameron-Martin space. In this paper we consider the equation

$$\lambda u - L^\Omega u = f \quad \text{in } \Omega, \quad (1)$$

where  $\lambda > 0$  and  $f \in L^2(\Omega, \gamma)$  are given,  $\Omega$  is an open convex set of  $X$  and  $L^\Omega$  is the Ornstein-Uhlenbeck operator associated to the quadratic form

$$\mathcal{E}_{\Omega, \gamma}(u, v) := \int_{\Omega} \langle \nabla_H u, \nabla_H v \rangle_H d\gamma \quad \text{for } u, v \in W^{1,2}(\Omega, \gamma).$$

Precise definition of the Sobolev spaces  $W^{1,2}(\Omega, \gamma)$ ,  $W^{2,2}(\Omega, \gamma)$ , and of the  $H$ -gradient  $\nabla_H$  are in the next section. As usual a function  $u \in W^{1,2}(\Omega, \gamma)$  is called weak solution to (1) if

$$\int_{\Omega} (\lambda u \varphi + \langle \nabla_H u, \nabla_H \varphi \rangle_H) d\gamma = \int_{\Omega} f \varphi d\gamma \quad \forall \varphi \in W^{1,2}(\Omega, \gamma).$$

It is not hard to see that for every  $\lambda > 0$  and  $f \in L^2(\Omega, \gamma)$ , problem (1) has a unique weak solution  $u$ . In this paper we prove a maximal regularity result for

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the weak solution  $u$  of (1), that is for every  $f \in L^2(\Omega, \gamma)$  the weak solution  $u$  belongs to  $W^{2,2}(\Omega, \gamma)$  and there exists  $C > 0$  independent of  $f$  such that

$$\|u\|_{W^{2,2}(\Omega, \gamma)} \leq C \|f\|_{L^2(\Omega, \gamma)}. \quad (2)$$

It is sufficient to have that (2) holds if  $F$  is a cylindrical smooth bounded function (see **Section 2**), because the space of such functions is dense in  $L^2(\Omega, \gamma)$ . In this case, we define a sequence of functions  $\{u_n\}_{n \in \mathbb{N}}$ , by using the cylindrical approximation  $\{\Omega_n\}_{n \in \mathbb{N}}$  of  $\Omega$  made in [8]. In particular,

$$u_n = \varphi_n \circ \pi_n$$

where  $\pi_n(X)$  is a finite dimensional subspace of  $H$ , identified in an obvious way with  $\mathbb{R}^q$  with  $q = q(n, f)$ . So  $\pi_n(\Omega_n)$  is identified with an open subset  $\mathcal{O}_n$  of  $\mathbb{R}^q$ , and  $\varphi_n : \mathcal{O}_n \subset \mathbb{R}^q \rightarrow \mathbb{R}$  solves

$$\begin{cases} \lambda \psi - L^{\mathcal{O}_n} \psi = \tilde{f} & \text{in } \mathcal{O}_n \subset \mathbb{R}^q, \\ \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial \mathcal{O}_n \end{cases} \quad (3)$$

where  $\tilde{f}$  is a suitable smooth bounded function. Here, the reference measure is the standard Gaussian measure  $N_{0,I}$ , and  $\nabla_H$  is the usual gradient. For the finite dimensional problems (3) we prove dimension free  $W^{2,2}$  estimates. Therefore the sequence  $\{u_n\}_{n \in \mathbb{N}}$  is bounded in  $W^{2,2}(\Omega, \gamma)$ , and a subsequence weakly converges to  $u \in W^{2,2}(\Omega, \gamma)$ . Eventually we prove that  $u$  is a weak solution of (1).

Moreover, under some regularity assumption on the boundary of  $\Omega$ , we prove that the weak solution of (1) satisfies

$$\langle \nabla_H u, \nabla_H g \rangle_H = 0 \quad (4)$$

on  $\partial\Omega$ , in the sense of traces. This identity plays the role of the Neumann boundary condition. We use the same sequence  $\{u_n\}_{n \in \mathbb{N}}$  defined above, and we show that

$$\int_{\Omega} (\lambda u_n - L^{\Omega_n} u_n) \varphi \, d\gamma = \int_{\Omega} f \varphi \, d\gamma,$$

for all smooth cylindrical functions  $\varphi$ , where  $L^{\Omega_n}$  is the Ornstein-Uhlenbeck operator associated to the quadratic form  $\mathcal{E}_{\Omega_n, \gamma}$ . Applying the integration by parts formula (6) we get

$$\int_{\Omega} \lambda \varphi u_n \, d\gamma + \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H \, d\gamma = \int_{\Omega} f \varphi \, d\gamma + \int_{\partial\Omega} \langle \nabla_H u_n, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho,$$

here  $g : X \rightarrow \mathbb{R}$  is a suitable convex function such that  $g^{-1}(0) = \partial\Omega$  and  $\rho$  is the surface measure associated to the Gaussian measure, see [7]. Taking the limit along a weakly convergent subsequence, we obtain

$$\int_{\Omega} \lambda \varphi u \, d\gamma + \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H \, d\gamma = \int_{\Omega} f \varphi \, d\gamma + \int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho,$$

for all smooth cylindrical functions  $\varphi$ . Since  $u$  is the weak solution of (1) then we can conclude that

$$\int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho = 0$$

for all smooth cylindrical functions  $\varphi$ , that is equivalent to (4).

The maximal  $L^p$  regularity for Ornstein-Uhlenbeck equations was established in [11] by Meyer when  $\Omega$  is the whole space  $X$  for  $1 < p < \infty$ . When  $\Omega \subsetneq X$  and  $p = 2$  the maximal regularity problem was also studied in Hilbert spaces by Da Prato and Lunardi in [5] with Dirichlet boundary condition and in [6] with Neumann boundary condition for a different class of differential operators that doesn't contain the classical Ornstein-Uhlenbeck operator. Also, the proof in [6] is different from ours because it uses a penalization method approaching the weak solution by a sequence of solutions of problems on whole  $X$ .

In finite dimension more results are available. Maximal  $L^p$  regularity, for  $p \in (1, \infty)$ , was studied by Metafune, Pruess, Rhandi, and Schnaubelt in [10] when  $\Omega = \mathbb{R}^n$  for a class of second order differential operators with unbounded coefficients that contains symmetric Ornstein-Uhlenbeck operators. Maximal  $L^2$  regularity in open convex sets of  $\mathbb{R}^n$ , with Neumann boundary condition, was established in [4] by methods different from ours.

## 2 Preliminaries and definitions

In this section we recall some basic definitions and notations. Hereafter  $h_i$  will denote the  $i$ -th element of an orthonormal basis of  $H$ ; for every  $k \in \mathbb{N}$  set  $\hat{h}_k = Q^{-1}(h_k)$  (see [1, p. 39-40]). If  $x_i \in \mathbb{R}^n$  we denote by  $D_i$  the directional derivative in the direction of  $x_i$  while by  $\partial_i$  we denote the directional derivative in the direction of  $h_i$ .

**Definition 1.**  $\mathcal{F}C_b^k(X)$  is the space of cylindrical functions of the form

$$f(x) = \varphi(l_1(x), \dots, l_n(x)),$$

with  $\varphi \in C_b^k(\mathbb{R}^n)$ ,  $l_i, \dots, l_n \in X^*$  and  $n \in \mathbb{N}$ .

**Definition 2.**  $W^{1,2}(\Omega, \gamma)$  is the Sobolev space defined as the completion of the restriction to  $\Omega$  of the elements of space  $\mathcal{F}C_b^1(X)$  with respect to the norm

$$\|f\|_{W^{1,2}(\Omega, \gamma)}^2 = \int_{\Omega} (f^2 + |\nabla_H f|_H^2) d\gamma.$$

where  $\nabla_H$  is the gradient along the direction of  $H$ .

**Definition 3.**  $W^{2,2}(\Omega, \gamma)$  is the Sobolev space defined as the completion of the restriction to  $\Omega$  of the elements of space  $\mathcal{F}C_b^2(X)$  with respect to the norm

$$\|f\|_{W^{2,2}(\Omega, \gamma)}^2 = \int_{\Omega} (f^2 + |\nabla_H f|_H^2 + \|D_H^2 f\|_{\mathcal{H}}^2) d\gamma.$$

where  $D_H^2$  is the  $H$ -Hessian operator and  $\|\cdot\|_{\mathcal{H}}$  is the Hilbert-Schmidt norm.

**Definition 4** (Weak solution). The function  $u \in W^{1,2}(\Omega, \gamma)$  is a weak solution of (1) if

$$\int_{\Omega} \lambda u \varphi d\gamma + \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega} f \varphi d\gamma \quad \forall \varphi \in \mathcal{F}C_b^1(X) \quad (5)$$

or equivalently for all  $\varphi \in W^{1,2}(\Omega, \gamma)$ .

**Assumption 1.** We suppose that  $\Omega = g^{-1}(-\infty, 0)$ , where  $g : X \rightarrow \mathbb{R}$  is a continuous function such that

- $g \in W^{2,p}(X, \gamma)$  for all  $p > 1$ ;
- there exists  $\delta > 0$  such that  $\frac{1}{|\nabla_H g|_H} \in L^p(g^{-1}(-\delta, \delta), \gamma)$  for all  $p > 1$ .

These conditions allow the definition of traces of Sobolev functions at  $g^{-1}(0) = \partial\Omega$ , see [3].

Let  $\varphi, \psi \in W^{1,2}(\Omega, \gamma)$ , we recall the integration by parts formula:

$$\int_{\Omega} \partial_k \varphi \psi \, d\gamma = - \int_{\Omega} \varphi \partial_k \psi \, d\gamma + \int_{\Omega} \varphi \psi \, \widehat{h}_k \, d\gamma + \int_{\partial\Omega} \text{Tr} \varphi \text{Tr} \psi \frac{\partial_k g}{|\nabla_H g|_H} \, d\rho. \quad (6)$$

where in the last integral  $\rho$  is the surface measure associated to the Gaussian measure and  $\text{Tr} \varphi, \text{Tr} \psi$  are the traces of the function  $\varphi, \psi$  (see [3]).

In [2] the *Logarithmic-Sobolev inequality* is proved:

$$\int_{\Omega} f^2 \log(f^2) \, d\gamma \leq \int_{\Omega} |\nabla_H f|_H^2 \, d\gamma + \|f\|_{L^2(\Omega, \gamma)}^2 \log(\|f\|_{L^2(\Omega, \gamma)}^2), \quad (7)$$

that holds for every  $f \in W^{1,2}(\Omega, \gamma)$ .

For  $u, v \in W^{1,2}(\Omega, \gamma)$  let

$$\mathcal{E}_{\Omega, \gamma}(u, v) := \int_{\Omega} \langle \nabla_H u, \nabla_H v \rangle_H \, d\gamma$$

be the quadratic form associated to  $\nabla_H$ ; we set

$$D(L^{\Omega}) = \left\{ u \in W^{1,2}(\Omega, \gamma) : \exists f \in L^2(\Omega, \gamma) \text{ s.t. } \begin{aligned} & \mathcal{E}_{\Omega, \gamma}(u, v) = - \int_{\Omega} f v \, d\gamma, \quad \forall v \in W^{1,2}(\Omega, \gamma) \end{aligned} \right\} \quad (8)$$

and we put  $L^{\Omega} u := f$ .

Let  $\mathcal{O}$  be a smooth convex set of  $\mathbb{R}^n$  and let  $\mu$  be the standard Gaussian measure in  $\mathbb{R}^n$ . Let  $L^{\mathcal{O}}$  be the Ornstein-Uhlenbeck operator associated to the quadratic form  $\mathcal{E}_{\mathcal{O}, \mu}$ . It is known, see [4], that

$$D(L^{\mathcal{O}}) = \left\{ f \in W^{2,2}(\mathcal{O}, \mu) : \langle x, \nabla f \rangle \in L^2(\mathcal{O}, \mu) \text{ and } \frac{\partial f}{\partial \nu} = 0 \right\} \quad (9)$$

where  $\nu(x)$  is the exterior normal vector to  $\partial\mathcal{O}$  at  $x$ . Moreover

$$L^{\mathcal{O}} f(x) = \Delta f(x) - \langle x, \nabla f(x) \rangle \text{ for every } f \in D(L^{\mathcal{O}}). \quad (10)$$

We recall the finite dimensional logarithmic Sobolev inequality

$$\int_{\mathcal{O}} f^2 \log(f^2) \, d\mu \leq \int_{\mathcal{O}} |\nabla_H f|_H^2 \, d\mu + \|f\|_{L^2(\mathcal{O}, \mu)}^2 \log(\|f\|_{L^2(\mathcal{O}, \mu)}^2), \quad (11)$$

that holds for each  $f \in W^{1,2}(\mathcal{O}, \mu)$ , see [4].

### 3 Finite-dimensional estimates

Let  $\mathcal{O}$  be an open smooth convex subset of  $\mathbb{R}^n$ , with fixed  $n$ . We assume that

$$\mathcal{O} = \{x \in \mathbb{R}^n : g(x) < 0\}$$

where  $g$  is a smooth convex function whose gradient does not vanish at the boundary  $\partial\mathcal{O}$ . We denote by  $\nu(x)$  the exterior normal vector to  $\partial\mathcal{O}$  at  $x$ ,  $\nu(x) = \frac{\nabla g(x)}{|\nabla g(x)|}$ . Let  $\mu$  be the standard Gaussian measure in  $\mathbb{R}^n$  and let  $L^\mathcal{O}$  be the associated Ornstein-Uhlenbeck operator, that is

$$L^\mathcal{O}\psi(x) = \sum_{i=1}^n D_{ii}\psi(x) - \sum_{i=1}^n x_i D_i\psi(x) \text{ for } \psi \in D(L^\mathcal{O}).$$

In this section we consider the following problem

$$\begin{cases} \lambda\psi - L^\mathcal{O}\psi = f & \text{in } \mathcal{O} \subset \mathbb{R}^n, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{O} \end{cases} \quad (12)$$

where  $f \in L^2(\mathcal{O}, \mu)$  and  $\lambda > 0$ .

Let us introduce a weighted surface measure on  $\partial\mathcal{O}$ :

$$d\sigma = N(x)dH^{n-1}$$

where  $N(x) = (2\pi)^{-n/2} \exp(-|x|^2/2)$  is the Gaussian weight and  $dH^{n-1}$  is the usual Hausdorff  $(n-1)$  dimensional measure. Using the surface measure  $d\sigma$  the integration by parts formula reads as:

$$\int_{\mathcal{O}} D_k \varphi \psi \, d\mu = - \int_{\mathcal{O}} \varphi D_k \psi \, d\mu + \int_{\mathcal{O}} x_k \varphi \psi \, d\mu + \int_{\partial\mathcal{O}} \frac{D_k g}{|\nabla g|} \varphi \psi \, d\sigma, \quad (13)$$

for each  $\varphi, \psi \in W^{1,2}(\mathcal{O}, \mu)$  one of which with bounded support, so the boundary integral is meaningful. Indeed  $W^{1,2}(\mathcal{O}, \mu) \subset W_{loc}^{1,2}(\mathcal{O}, dx)$  and the trace at the boundary of any function in  $W^{1,2}(\mathcal{O}, \mu)$  belongs to  $L_{loc}^2(\partial\mathcal{O}, dH^{n-1}) = L_{loc}^2(\partial\mathcal{O}, d\sigma)$ .

Applying (13) with  $\varphi$  replaced by  $D_k \varphi$  and summing up, we find

$$\int_{\mathcal{O}} L^\mathcal{O} \varphi \psi \, d\mu = - \int_{\mathcal{O}} \langle \nabla \varphi, \nabla \psi \rangle d\mu + \int_{\partial\mathcal{O}} \frac{\langle \nabla \varphi, \nabla g \rangle}{|\nabla g|} \psi \, d\sigma \quad (14)$$

for every  $\varphi \in W^{2,2}(\mathcal{O}, \mu)$ ,  $\psi \in W^{1,2}(\mathcal{O}, \mu)$  one of which with bounded support.

Now we give dimension free estimates for the weak solution  $u \in W^{1,2}(\mathcal{O}, \mu)$  to (12) with  $\lambda > 0$  and  $f \in L^2(\mathcal{O}, \mu)$ . We can suppose  $f \in C_c^\infty(\mathcal{O})$  because  $C_c^\infty(\mathcal{O})$  is dense in  $L^2(\mathcal{O}, \mu)$ . In this case, thanks to classical results about elliptic equations with smooth coefficients we know that the weak solution  $u$  of (12) belongs to  $C^\infty(\overline{\mathcal{O}}) \subset W_{loc}^{2,2}(\mathcal{O}, \mu)$ . Since  $\mathcal{O}$  can be unbounded, we introduce a smooth cutoff function  $\theta : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$0 \leq \theta(x) \leq 1, \quad |\nabla \theta(x)| \leq 2 \quad \forall x \in \mathbb{R}^n, \quad \theta \equiv 1 \text{ in } B(0, 1), \quad \theta \equiv 0 \text{ outside } B(0, 2)$$

and we set, for  $R > 0$

$$\theta_R(x) = \theta(x/R), \quad x \in \mathbb{R}^n.$$

For the  $W^{1,2}$  estimates we take  $u$  as a test function in the definition of weak solution and we get

$$\lambda \int_{\mathcal{O}} u^2 d\mu + \int_{\mathcal{O}} |\nabla u|^2 d\mu = \int_{\mathcal{O}} f u d\mu, \quad (15)$$

then

$$\int_{\mathcal{O}} u^2 d\mu \leq \frac{1}{\lambda^2} \|f\|_{L^2(\mathcal{O}, \mu)}^2, \quad \int_{\mathcal{O}} |\nabla u|^2 d\mu \leq \frac{1}{\lambda} \|f\|_{L^2(\mathcal{O}, \mu)}^2. \quad (16)$$

The following lemma takes into the account the convexity of  $\mathcal{O}$ .

**Lemma 1.** *If  $u \in C^2(\overline{\mathcal{O}})$  satisfies  $\langle \nabla u, \nu \rangle = 0$  on  $\partial\mathcal{O}$  then*

$$\langle D^2 u \cdot \nabla u, \nu \rangle \leq 0 \text{ on } \partial\mathcal{O}.$$

*Proof.* We recall that  $\partial\mathcal{O} = g^{-1}(0)$  where  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth convex function. Let  $\tau \in \mathbb{R}^n$  such that  $\langle \tau, \nu(x) \rangle = 0$  for  $x \in \partial\mathcal{O}$ , then we have

$$\left\langle \frac{\partial \nu}{\partial \tau}(x), \tau \right\rangle \geq 0 \quad \forall x \in \partial\mathcal{O}. \quad (17)$$

Indeed

$$\begin{aligned} \frac{\partial \nu}{\partial \tau} &= \frac{\partial}{\partial \tau} \left( \frac{\nabla g}{|\nabla g|} \right) = \frac{1}{|\nabla g|} \frac{\partial}{\partial \tau} (\nabla g) + \frac{\partial}{\partial \tau} \left( \frac{1}{|\nabla g|} \right) \nabla g \\ &= \frac{1}{|\nabla g|} D^2 g \cdot \tau + \left\langle \nabla \left( \frac{1}{|\nabla g|} \right), \tau \right\rangle \nabla g, \end{aligned}$$

then for  $x \in \partial\mathcal{O}$  we have

$$\begin{aligned} \left\langle \frac{\partial \nu}{\partial \tau}(x), \tau \right\rangle &= \frac{1}{|\nabla g(x)|} \langle D^2 g(x) \cdot \tau, \tau \rangle + \left\langle \nabla \left( \frac{1}{|\nabla g(x)|} \right), \tau \right\rangle \langle \nabla g(x), \tau \rangle \\ &= \frac{1}{|\nabla g(x)|} \langle D^2 g(x) \cdot \tau, \tau \rangle \geq 0 \end{aligned}$$

since  $D^2 g$  is a positive semi-definite symmetric matrix. Now we recall that  $\langle \nabla u, \nu \rangle = 0$  on  $\partial\mathcal{O}$  therefore

$$\frac{\partial}{\partial \tau} (\langle \nabla u(x), \nu(x) \rangle) = \langle D^2 u(x) \tau, \nu(x) \rangle + \langle \nabla u(x), \frac{\partial \nu}{\partial \tau}(x) \rangle = 0, \quad x \in \partial\mathcal{O}$$

for each  $\tau \in \mathbb{R}^n$  such that  $\langle \tau, \nu \rangle = 0$  on  $\partial\mathcal{O}$ . If we take  $\tau = \nabla u(x)$  then we get

$$\langle D^2 u(x) \cdot \nabla u(x), \nu(x) \rangle = - \left\langle \tau, \frac{\partial \nu}{\partial \tau}(x) \right\rangle \leq 0, \quad x \in \partial\mathcal{O}.$$

□

Now we can give an estimate of the second order derivatives of  $u$ .

**Proposition 1.** *For every  $f \in C_c^\infty(\mathcal{O})$  and  $\varepsilon > 0$  there exists  $R_0 > 0$  such that for  $R > R_0$  the solution  $u$  to (12) satisfies*

$$(1 - \varepsilon) \int_{\mathcal{O}} \theta_R^2 \text{Tr}[(D^2 u)^2] d\mu \leq \left(2 + \frac{\varepsilon}{\lambda}\right) \|f\|_{L^2(\mathcal{O}, \mu)}^2. \quad (18)$$

*Proof.* Recall that  $u \in C^\infty(\overline{\mathcal{O}})$ ; differentiating (12) with respect to  $x_h$  yields

$$\lambda D_h u - \Delta D_h u - \langle x, \nabla(D_h u) \rangle + D_h u = D_h f.$$

Multiplying by  $D_h u \theta_R^2$  we obtain

$$(\lambda + 1) (D_h u)^2 \theta_R^2 - \Delta D_h u \cdot D_h u \theta_R^2 - \langle x, \nabla(D_h u) \rangle D_h u \theta_R^2 = D_h f D_h u \theta_R^2.$$

Integrating over  $\mathcal{O}$  and using (14) yields

$$\begin{aligned} & \int_{\mathcal{O}} (\lambda + 1) (D_h u)^2 \theta_R^2 d\mu + \int_{\mathcal{O}} |\nabla(D_h u)|^2 \theta_R^2 d\mu + 2 \int_{\mathcal{O}} \theta_R \langle \nabla(D_h u), \nabla \theta_R \rangle D_h u d\mu \\ &= \int_{\partial \mathcal{O}} \frac{\langle \nabla(D_h u), \nabla g \rangle D_h u}{|\nabla g|} \theta_R^2 d\sigma + \int_{\mathcal{O}} D_h f D_h u \theta_R^2 d\mu. \end{aligned}$$

Summing over  $h$  we obtain

$$\begin{aligned} & \int_{\mathcal{O}} (\lambda + 1) |\nabla u|^2 \theta_R^2 d\mu + \int_{\mathcal{O}} \text{Tr}[(D^2 u)^2] \theta_R^2 d\mu + 2 \int_{\mathcal{O}} \langle D^2 u \cdot \nabla u, \nabla \theta_R \rangle \theta_R d\mu \\ &= \int_{\partial \mathcal{O}} \frac{\langle D^2 u \cdot \nabla u, \nabla g \rangle}{|\nabla g|} \theta_R^2 d\sigma + \int_{\mathcal{O}} \langle \nabla f, \nabla u \rangle \theta_R^2 d\mu. \end{aligned}$$

Since  $f$  has compact support, for  $R$  large enough  $\theta_R \equiv 1$  on the support of  $f$ . For such  $R$  we obtain

$$\left| \int_{\mathcal{O}} \langle \nabla f, \nabla u \rangle \theta_R^2 d\mu \right| = \left| - \int_{\mathcal{O}} L^{\mathcal{O}} u f d\mu \right| = \left| \int_{\mathcal{O}} (\lambda u - f) f d\mu \right| \leq 2 \|f\|_{L^2(\mathcal{O}, \mu)}^2.$$

Moreover

$$\begin{aligned} & \left| \int_{\mathcal{O}} \langle D^2 u \nabla u, \nabla \theta_R \rangle \theta_R d\mu \right| \leq \int_{\mathcal{O}} \sum_{i,j=1}^n |D_{ij} u| |D_j u| |D_i \theta_R| \theta_R d\mu \\ & \leq \frac{1}{2} \int_{\mathcal{O}} \sum_{i,j=1}^n |D_{ij} u|^2 |D_i \theta_R| \theta_R^2 d\mu + \frac{1}{2} \int_{\mathcal{O}} \sum_{i,j=1}^n |D_j u|^2 |D_i \theta_R| d\mu \\ & \leq \frac{1}{2} \frac{\|\nabla \theta\|_\infty}{R} \int_{\mathcal{O}} \theta_R^2 \text{Tr}[(D^2 u)^2] d\mu + \frac{1}{2} \frac{\|\nabla \theta\|_\infty}{R} \int_{\mathcal{O}} |\nabla u|^2 d\mu \\ & \leq \left( \frac{1}{2} \int_{\mathcal{O}} \text{Tr}[(D^2 u)^2] \theta_R^2 d\mu + \frac{1}{2\lambda} \|f\|_{L^2(\mathcal{O}, \mu)}^2 \right) \frac{\|\nabla \theta\|_\infty}{R}. \end{aligned}$$

Taking  $R$  large enough, such that  $\|\nabla \theta\|_\infty / R \leq \varepsilon$ , we get

$$(1 - \varepsilon) \int_{\mathcal{O}} \theta_R^2 \text{Tr}[(D^2 u)^2] d\mu \leq \left(2 + \frac{\varepsilon}{\lambda}\right) \|f\|_{L^2(\mathcal{O}, \mu)}^2 + \int_{\partial \mathcal{O}} \theta_R^2 \frac{\langle D^2 u \cdot \nabla u, \nabla g \rangle}{|\nabla g|} d\sigma.$$

By using **Lemma 1**, the statement follows.  $\square$

**Theorem 1.** For each  $\lambda > 0$  there exists  $C = C(\lambda) > 0$ , independent of  $n$  and  $\mathcal{O}$ , such that for each  $f \in L^2(\mathcal{O}, \mu)$  the weak solution  $u$  of problem (12) belongs to  $W^{2,2}(\mathcal{O}, \mu)$ , and satisfies

$$\|u\|_{W^{2,2}(\mathcal{O}, \mu)} \leq C \|f\|_{L^2(\mathcal{O}, \mu)}. \quad (19)$$

*Proof.* Let  $f \in C_c^\infty(\mathcal{O})$ . Taking the limit as  $R \rightarrow \infty$  in (18) and using the monotone convergence theorem, we get

$$(1 - \varepsilon) \int_{\mathcal{O}} \text{Tr}[(D^2 u)^2] d\mu \leq \left(2 + \frac{\varepsilon}{\lambda}\right) \|f\|_{L^2(\mathcal{O}, \mu)}^2.$$

Now taking the limit as  $\varepsilon \rightarrow 0$  we get

$$\int_{\mathcal{O}} \text{Tr}[(D^2 u)^2] d\mu \leq 2 \|f\|_{L^2(\mathcal{O}, \mu)}^2. \quad (20)$$

Taking into account (15), (16), and (20) we obtain

$$\begin{aligned} \|u\|_{W^{2,2}(\mathcal{O}, \mu)}^2 &= \|u\|_{L^2(\mathcal{O}, \mu)} + \|\nabla u\|_{L^2(\mathcal{O}, \mu)} + \|\text{Tr}[(D^2 u)^2]\|_{L^2(\mathcal{O}, \mu)} \\ &\leq \left(\frac{1}{\lambda^2} + \frac{1}{\lambda} + 2\right) \|f\|_{L^2(\mathcal{O}, \mu)}^2 \end{aligned}$$

which is (19) with  $C(\lambda) = \frac{1}{\lambda^2} + \frac{1}{\lambda} + 2$ . For  $f \in L^2(\mathcal{O}, \mu)$  the statement follows approaching it by a sequence of functions belonging to  $C_c^\infty(\mathcal{O})$ .  $\square$

Now we get a characterization of the domain of  $L^{\mathcal{O}}$ . We recall that (9) holds, and we prove that the condition  $\langle \cdot, \nabla f \rangle \in L^2(\mathcal{O}, \mu)$  can be omitted.

**Proposition 2.** *If  $f \in W^{2,2}(\mathcal{O}, \mu)$  then  $\langle x, \nabla f \rangle \in L^2(\mathcal{O}, \mu)$ , moreover the map*

$$f \mapsto \langle \cdot, \nabla f \rangle$$

*is continuous from  $W^{2,2}(\mathcal{O}, \mu)$  to  $L^2(\mathcal{O}, \mu)$ .*

*Proof.* Let  $f \in W^{2,2}(\mathcal{O}, \mu)$ , then

$$\int_{\mathcal{O}} |\langle \nabla f, x \rangle|^2 d\mu = \int_{\mathcal{O}} \sum_{i=1}^n (D_i f x_i)^2 d\mu$$

by assumption  $D_i f \in W^{1,2}(\mathcal{O}, \mu)$  and if  $c < 1/4$ , by using (11), we have

$$\begin{aligned} &\int_{\mathcal{O}} (D_i f(x))^2 x_i^2 e^{-|x|^2/2} dx \\ &= \int_{\{x \in \mathcal{O}: cx_i^2 > \log |D_i f(x)|\}} (D_i f(x))^2 x_i^2 e^{-|x|^2/2} dx \\ &\quad + \int_{\{x \in \mathcal{O}: cx_i^2 \leq \log |D_i f(x)|\}} (D_i f(x))^2 x_i^2 e^{-|x|^2/2} dx \\ &\leq \int_{\mathcal{O}} e^{2cx_i^2} x_i^2 e^{-|x|^2/2} dx + \int_{\mathcal{O}} \frac{1}{c} |D_i f|^2 \log |D_i f| e^{-|x|^2/2} dx \\ &\leq C_1 + \frac{1}{c} \left( \int_{\mathcal{O}} |\nabla D_i f| d\mu + \frac{1}{2} \int_{\mathcal{O}} (D_i f)^2 d\mu \log \left( \int_{\mathcal{O}} (D_i f)^2 d\mu \right) \right). \end{aligned}$$

Summing over  $i$  from 1 to  $n$  we have  $\langle \nabla f, x \rangle \in L^2(\mathcal{O}, \mu)$ .  $\square$



## 4 Approximation by cylindrical functions

Now we consider the infinite dimensional problem. Let  $\Omega \subset X$  be an open convex set and let  $\{\Omega_n\}$  be a sequence of cylindrical open convex sets, defined in [8], of the form  $\Omega_n = \pi_n^{-1}(\mathcal{O}_n)$  where  $\mathcal{O}_n \subset F_n$ ,  $F_n$  is a finite dimensional subspace of  $Q(X^*) \subset H$  with  $\dim F_n = j(n) \leq n$ ,  $F_n \subset F_{n+1}$  for  $n \in \mathbb{N}$ , and  $\pi_n : X \rightarrow F_n$  is the projection defined below. Let  $\{h_n\}_{n \in \mathbb{N}} \subset Q(X^*)$  be an orthonormal basis of the Cameron-Martin space  $H$  such that  $F_n = \text{span}\{h_1, \dots, h_{j(n)}\}$ . Therefore

$$\pi_n(x) = \sum_{i=1}^{j(n)} \widehat{h}_i(x) h_i.$$

Moreover  $\Omega_{n+1} \subset \Omega_n$ ,  $\partial \mathcal{O}_n$  is smooth,  $\Omega \subset \Omega_n$  and

$$\overline{\Omega} = \bigcap_{n \in \mathbb{N}} \overline{\Omega}_n, \quad \gamma \left( \bigcap_{n \in \mathbb{N}} \Omega_n \setminus \Omega \right) = 0.$$

We recall that since  $\Omega$  and  $\Omega_n$  are open convex sets, then  $\gamma(\partial \Omega) = \gamma(\partial \Omega_n) = 0$ .

Now we show that the restriction to  $\Omega$  of cylindrical continuous smooth functions is dense in  $L^2(\Omega, \gamma)$ . Let  $\psi \in L^2(\Omega, \gamma)$ , then the zero extension outside  $\Omega$ ,  $\bar{\psi}$ , belongs to  $L^2(X, \gamma)$ . We have from [1, **Corollary 3.5.2**] that there exists a sequence of  $L^2$  cylindrical functions  $\psi_n$  that converges to  $\bar{\psi}$  in  $L^2(X, \gamma)$ . In its turn, each  $\psi_n$  may be approached by a sequence of  $C_b^\infty$  functions.

Therefore we suppose  $f \in \mathcal{FC}_b^\infty(X)$ . Then, for some  $k \in \mathbb{N}$ ,

$$f(x) = w(l_1(x), \dots, l_k(x))$$

where  $w \in C_b^\infty(\mathbb{R}^k)$ ,  $l_i \in X^*$  for  $i = 1, \dots, k$ .

Let  $G = G(n, f) := \text{span}\{F_n, Q(l_1), \dots, Q(l_k)\}$ . Then  $G$  is a subspace of  $H$  of dimension  $q = q(n, f) \leq j(n) + k$ ; setting  $d := q - j(n)$  let  $\mathcal{O} = \mathcal{O}(n, f) := \mathcal{O}_n \times \mathbb{R}^d$ . If we denote by

$$\pi_G(x) = \sum_{i=1}^q \widehat{h}_i(x) h_i$$

then

$$f(x) = \tilde{f}(\pi_G(x))$$

where  $\tilde{f} \in C_b^\infty(G)$ . Let  $\gamma_G$  be the induced measure  $\gamma \circ \pi_G^{-1}$  in  $G$ ; if  $G$  is identified with  $\mathbb{R}^q$  through the isomorphism  $x \mapsto (\widehat{h}_1(x), \dots, \widehat{h}_q(x))$  then  $\gamma_G$  is the standard Gaussian measure in  $\mathbb{R}^q$ .

We recall that  $L^{\Omega_n}$  is the Ornstein-Uhlenbeck operator associated to the quadratic form  $\mathcal{E}_{\Omega_n, \gamma}$  while  $L^{\mathcal{O}}$  is Ornstein-Uhlenbeck operator associated to the quadratic form  $\mathcal{E}_{\mathcal{O}, \gamma_G}$ .

**Proposition 3.** *Let  $v$  be the weak solution of the finite dimensional problem*

$$\lambda v - L^{\mathcal{O}} v = \tilde{f}|_{\mathcal{O}} \quad \text{in } \mathcal{O}$$

*Then  $u(x) := v(\pi_G(x))$  is the weak solution of*

$$\lambda u - L^{\Omega_n} u = f|_{\Omega_n} \quad \text{in } \Omega_n$$

*Proof.* We remark that the space  $X$  can be split as  $X = G \times \tilde{X}$  where  $\tilde{X} = (I - \pi_G)(X)$ , and  $\gamma = \gamma_G \otimes \tilde{\gamma}$  where  $\tilde{\gamma} = \gamma \circ (I - \pi_G)^{-1}$  is the measure induced on  $\tilde{X}$  by the projection  $I - \pi_G$ . Let  $\varphi \in W^{1,2}(\Omega_n, \gamma)$ , then

$$\begin{aligned}
& \int_{\Omega_n} (\lambda u(x)\varphi(x) + \langle \nabla_H u(x), \nabla_H \varphi(x) \rangle_H) \gamma(dx) \\
&= \int_{\Omega_n} \lambda v(\pi_G(x))\varphi(\pi_G(x) + (I - \pi_G)(x)) \\
&\quad + \langle \nabla_H v(\pi_G(x)), \nabla_H \varphi(\pi_G(x) + (I - \pi_G)(x)) \rangle_H \gamma(dx) \\
&= \int_{\mathcal{O} \times \tilde{X}} \lambda v(\xi)\tilde{\varphi}(\xi + y) \\
&\quad + \langle \nabla v(\xi), \nabla \tilde{\varphi}(\xi + y) \rangle_{\gamma_G(d\xi)\tilde{\gamma}(dy)} \quad (\text{where } \tilde{\varphi}(\cdot + y) \in W^{1,2}(\mathcal{O}, \gamma_G)) \\
&= \int_{\mathcal{O} \times \tilde{X}} \tilde{f}(\xi)\tilde{\varphi}(\xi + y)\gamma_G(d\xi)\tilde{\gamma}(dy) \\
&= \int_{\Omega_n} \tilde{f}(\pi_G(x))\varphi(x)\gamma(dx) \\
&= \int_{\Omega_n} f(x)\varphi(x)\gamma(dx),
\end{aligned}$$

and the statement follows.  $\square$

**Proposition 4.** *The function  $u$  satisfies*

$$\|u\|_{W^{2,2}(\Omega, \gamma)} \leq K$$

where

$$K := C\|f\|_{L^2(\Omega_1, \gamma)}$$

and  $C$  is the constant of **Theorem 1**.

*Proof.* We recall that  $u(x) = v(\pi_G(x))$ . Then

$$\begin{aligned}
\|u\|_{W^{2,2}(\Omega, \gamma)}^2 &\leq \|u\|_{W^{2,2}(\Omega_n, \gamma)}^2 \\
&= \int_{\Omega_n} |u(x)|^2 + \sum_{i=1}^{\infty} |D_i u(x)|^2 + \sum_{i,j=1}^{\infty} |D_{ij} u(x)|^2 \gamma(dx) \\
&= \int_{\mathcal{O}} \left( |v(\xi)|^2 + \sum_{i=1}^q |\partial_i v(\xi)|^2 \right. \\
&\quad \left. + \sum_{i,j=1}^q |\partial_{ij} v(\xi)|^2 \right) \mu(d\xi) \quad (\text{By using Theorem 1}) \\
&\leq C^2 \|\tilde{f}\|_{L^2(\mathcal{O}, \mu)}^2 = C^2 \|f\|_{L^2(\Omega_n, \gamma)}^2 \leq C^2 \|f\|_{L^2(\Omega_1, \gamma)}^2
\end{aligned}$$

$\square$

If we consider the sequence  $\{u_n\}_{n \in \mathbb{N}}$  of weak solutions of the problems

$$\lambda \psi - L^{\Omega_n} \psi = f|_{\Omega_n} \quad \text{in } \Omega_n.$$

By **Proposition 4** it follows

$$\|u_n\|_{W^{2,2}(\Omega,\gamma)} \leq K.$$

Possibly replacing  $u_n$  by a subsequence, there exists  $u \in W^{2,2}(\Omega,\gamma)$  such that  $u_n \rightharpoonup u$  in  $W^{2,2}(\Omega,\gamma)$ .

**Proposition 5.** *The function  $u$  is the weak solution of (1).*

*Proof.* We know that for all  $\varphi \in \mathcal{FC}_b^1(X)$

$$\int_{\Omega_n} \lambda u_n \varphi \, d\gamma + \int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega_n} f \varphi \, d\gamma.$$

We claim that

$$\lim_{n \rightarrow \infty} \int_{\Omega_n} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u \varphi \, d\gamma.$$

Indeed,

$$\int_{\Omega_n} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u_n \varphi \, d\gamma + \int_{\Omega \setminus \Omega_n} \lambda u_n \varphi \, d\gamma; \quad (21)$$

by the weak convergence

$$\lim_{n \rightarrow \infty} \int_{\Omega} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u \varphi \, d\gamma$$

while

$$\left| \int_{\Omega \setminus \Omega_n} \lambda u_n \varphi \, d\gamma \right| \leq \lambda \left( \int_{\Omega \setminus \Omega_n} |u_n|^2 d\gamma \right)^{1/2} \left( \int_{\Omega \setminus \Omega_n} |\varphi|^2 d\gamma \right)^{1/2} \leq \lambda K \left( \int_{\Omega \setminus \Omega_n} |\varphi|^2 d\gamma \right)^{1/2}$$

that goes to zero as  $n \rightarrow \infty$  by the absolute continuity of the integral. Now we claim that

$$\lim_{n \rightarrow \infty} \int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H d\gamma.$$

In fact,

$$\int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma + \int_{\Omega \setminus \Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma.$$

By the weak convergence in  $W^{1,2}(\Omega,\gamma)$

$$\lim_{n \rightarrow \infty} \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H d\gamma$$

while

$$\begin{aligned} \left| \int_{\Omega \setminus \Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma \right| &\leq \lambda \left( \int_{\Omega \setminus \Omega_n} |\nabla_H u_n|_H^2 d\gamma \right)^{1/2} \left( \int_{\Omega \setminus \Omega_n} |\nabla_H \varphi|_H^2 d\gamma \right)^{1/2} \\ &\leq \lambda K \left( \int_{\Omega \setminus \Omega_n} |\nabla_H \varphi|_H^2 d\gamma \right)^{1/2} \end{aligned}$$

that goes to zero as  $n \rightarrow \infty$ .

Moreover,

$$\lim_{n \rightarrow \infty} \int_{\Omega_n} f \varphi \, d\gamma = \int_{\Omega} f \varphi \, d\gamma.$$

Therefore letting  $n \rightarrow \infty$  in (21) we get that  $u$  satisfies (5).  $\square$

Finally we give the maximal regularity estimate.

**Theorem 2.** *If  $u$  is the weak solution of  $\lambda u - L^\Omega u = f$  on  $\Omega$  then  $u \in W^{2,2}(\Omega, \gamma)$  and*

$$\|u\|_{W^{2,2}(\Omega, \gamma)} \leq C \|f\|_{L^2(\Omega, \gamma)}$$

*Proof.* By **Proposition 4** it follows

$$\|u_n\|_{W^{2,2}(\Omega, \gamma)} \leq C \|f\|_{L^2(\Omega_n, \gamma)} \quad (22)$$

where  $C = C(\lambda)$  is the constant of the **Theorem 1**.

We remark that

$$\lim_{n \rightarrow \infty} \|f\|_{L^2(\Omega_n, \gamma)} = \|f\|_{L^2(\Omega, \gamma)}$$

since  $\gamma(\Omega_n \setminus \Omega) \rightarrow 0$ .

By the weak convergence of  $u_n$  to  $u$  we have

$$\|u\|_{W^{2,2}(\Omega, \gamma)} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{W^{2,2}(\Omega, \gamma)}.$$

Letting  $n \rightarrow \infty$  in (22) we get our claim.  $\square$

## 5 The Neumann boundary condition

In this section we put under **Assumption 1** and we prove that the weak solution  $u$  of (1) satisfies a Neumann type boundary condition.

First we prove a useful lemma.

**Proposition 6.** *If  $u \in L^p(\partial\Omega, \rho)$  and*

$$\int_{\partial\Omega} u \varphi \, d\rho = 0 \quad \forall \varphi \in \mathcal{FC}_b^1(X),$$

*then  $u = 0$   $\rho$ -a.e. in  $\partial\Omega$ .*

*Proof.* Since the map

$$v \mapsto \int_{\partial\Omega} uv \, d\rho$$

is continuous from  $W^{1,q}(\Omega, \gamma)$  to  $\mathbb{R}$  for all  $q > p'$ , and  $\mathcal{FC}_b^1(X)$  is dense in  $W^{1,q}(\Omega, \gamma)$ , it follows that

$$\int_{\partial\Omega} u \psi \, d\rho = 0 \quad \forall \psi \in W^{1,q}(\Omega, \gamma).$$

In particular, since the restrictions to  $\Omega$  of the Lipschitz continuous and bounded functions  $\psi : X \rightarrow \mathbb{R}$  belong to  $W^{1,q}(\Omega, \gamma)$ , we have

$$\int_{\partial\Omega} u \psi \, d\rho = 0 \quad \forall \psi \in Lip_b(X).$$

**Lemma 3** yields

$$\int_{\partial\Omega} u\psi \, d\rho = 0 \quad \forall \psi \in L^q(\partial\Omega, \rho)$$

and this implies that  $u = 0$   $\rho$ -a.e..  $\square$

Now we are ready to prove that the weak solution of (1) satisfies a boundary condition similar to the Neumann boundary condition.

**Proposition 7.** *If  $u$  is the weak solution of  $\lambda u - Lu = f$  on  $\Omega$  then*

$$\langle \nabla_H u(x), \frac{\nabla_H g(x)}{|\nabla_H g(x)|_H} \rangle_H = 0 \quad \rho - a.e \, x \in \partial\Omega. \quad (23)$$

*Proof.* We fix  $\varphi \in \mathcal{FC}_b^1(X)$ . We denote by  $u_n$  the solution to

$$\lambda\psi - L^{\Omega_n}\psi = f|_{\Omega_n} \quad \text{in } \Omega_n. \quad (24)$$

We recall that  $u_n$  is a cylindrical function and, thanks to the result of **Section 4**, we have  $u_n \in W^{2,2}(\Omega_n, \gamma)$ . We multiply the differential equation (24) by  $\varphi$  and we integrate on  $\Omega$ , getting

$$\int_{\Omega} (\lambda u_n - L^{\Omega_n} u_n) \varphi \, d\gamma = \int_{\Omega} f \varphi \, d\gamma.$$

We recall that  $L^{\Omega_n} u_n$  is cylindrical, then

$$L^{\Omega_n} u_n(x) = \sum_{i=1}^q \partial_{ii} u_n(x) - \hat{h}_i(x) \partial_i u_n(x).$$

Therefore, by using (6), we obtain

$$\int_{\Omega} \lambda \varphi u_n \, d\gamma + \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega} f \varphi \, d\gamma + \int_{\partial\Omega} \langle \nabla_H u_n, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho,$$

where

$$\langle \nabla_H u_n, \nabla_H \varphi \rangle_H = \sum_{i=1}^q \partial_i u_n \partial_i \varphi,$$

and

$$\langle \nabla_H u_n, \nabla_H g \rangle_H = \sum_{i=1}^q \partial_i u_n \partial_i g.$$

As in the previous section we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \lambda \varphi u_n \, d\gamma = \int_{\Omega} \lambda \varphi u \, d\gamma,$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H d\gamma,$$

We claim that the map

$$v \mapsto \int_{\partial\Omega} \langle \nabla_H v, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho$$

from  $W^{2,2}(\Omega, \gamma)$  to  $\mathbb{R}$  belongs to  $(W^{2,2}(\Omega, \gamma))'$ . Indeed, the function

$$x \mapsto \langle \nabla_H v(x), \frac{\nabla_H g(x)}{|\nabla_H g(x)|_H} \rangle_H \varphi(x) =: F(x)$$

belongs to  $W^{1,q}(\Omega, \gamma)$  for all  $q \in (1, 2)$ . Moreover  $\|F\|_{W^{1,q}(\Omega, \gamma)} \leq \tilde{C}\|v\|_{W^{2,2}(\Omega, \gamma)}$ , and the trace operator is linear and continuous from  $W^{1,q}(\Omega, \gamma)$  to  $L^1(\partial\Omega, \rho)$ . Therefore, since  $u_n \rightharpoonup u$  in  $W^{2,2}(\Omega, \gamma)$ ,

$$\lim_{n \rightarrow \infty} \int_{\partial\Omega} \langle \nabla_H u_n, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho = \int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho.$$

Then we have

$$\int_{\Omega} \lambda u \varphi \, d\gamma + \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega} f \varphi \, d\gamma + \int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho$$

and since  $u$  is a weak solution of (1) we get

$$\int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho = 0$$

for all  $\varphi \in \mathcal{FC}_b^1(X)$ . By using **Proposition 6** we obtain (23).  $\square$

Therefore, if  $u \in D(L)$  then  $u \in W^{2,2}(\Omega, \gamma)$  and  $u$  satisfies the Neumann boundary condition (23).

## A Density properties

In this appendix we show some density results for which we thank Simone Ferrari. Let  $(Y, d)$  be a complete metric space and let  $\rho$  be a finite Radon measure defined on the Borel sets of  $Y$ . Let  $BUC(Y)$  be the set of real value uniformly bounded continuous functions and let  $Lip_b(Y)$  be the set of Lipschitz bounded functions.

**Lemma 2.** *Let  $f : Y \rightarrow \mathbb{R}$  be a bounded  $\rho$ -measurable function. Then for all  $\varepsilon > 0$  there exists  $g \in BUC(Y)$  such that*

$$\rho(\{x \in Y : f(x) \neq g(x)\}) < \varepsilon$$

and

$$\sup_{x \in Y} |g(x)| \leq 2 \sup_{x \in Y} |f(x)|.$$

*Proof.* We fix  $\varepsilon > 0$ . Since  $\rho$  is a Radon measure then there exists  $K_0$ , compact subset of  $Y$ , such that  $\rho(Y \setminus K_0) < \varepsilon$ . By the Lusin theorem there exists a function  $f_0 \in C(K_0) = BUC(K_0)$  such that:

$$\rho(\{x \in K_0 : f_0(x) \neq f|_{K_0}(x)\}) < \varepsilon$$

and

$$\sup_{x \in K_0} |f_0(x)| \leq \sup_{x \in K_0} |f(x)| \leq \sup_{x \in Y} |f(x)|.$$

We define the following function, studied in [9],

$$g(x) = \begin{cases} f(x) & \text{if } x \in K_0 \\ \inf_{y \in K_0} f_0(y) \frac{d(x, y)}{d(x, K_0)} & \text{if } x \notin K_0 \end{cases}$$

then  $g$  is a  $BUC$  extension of  $f_0$  to the whole  $Y$ . We remark that for  $x \notin K_0$  there exists  $y_\varepsilon \in K_0$  such that

$$d(x, K_0) = \inf_{y \in K_0} d(x, y) \geq d(x, y_\varepsilon) - \varepsilon,$$

therefore for  $x \notin K_0$  we have

$$|g(x)| = \left| \inf_{y \in K_0} f_0(y) \frac{d(x, y)}{d(x, K_0)} \right| \leq \sup_{x \in Y} |f(x)| \frac{d(x, y_\varepsilon)}{d(x, K_0)} \leq \sup_{x \in Y} |f(x)| \frac{d(x, K_0) + \varepsilon}{d(x, K_0)}$$

for all  $\varepsilon$ . Then for all  $x \notin K_0$  we have

$$g(x) \leq \sup_{y \in Y} |f(y)|.$$

Finally

$$\begin{aligned} \sup_{x \in Y} |g(x)| &= \sup_{x \in Y} |g|_{K_0}(x) + g|_{Y \setminus K_0}(x) \leq \sup_{x \in K_0} |g(x)| + \sup_{x \in Y \setminus K_0} |g(x)| \\ &= \sup_{x \in K_0} |f_0(x)| + \sup_{x \in Y \setminus K_0} |g(x)| \leq 2 \sup_{x \in Y} |f(x)|. \end{aligned}$$

Moreover

$$\begin{aligned} \rho(\{x \in Y : g(x) \neq f(x)\}) &\leq \rho(\{x \in K_0 : g(x) \neq f(x)\}) + \rho(\{x \in Y \setminus K_0 : g(x) \neq f(x)\}) \\ &\leq \rho(\{x \in K_0 : f_0(x) \neq f(x)\}) + \rho(Y \setminus K_0) < 2\varepsilon. \end{aligned}$$

□

**Lemma 3.** *The subspace  $Lip_b(Y)$  is dense in  $L^p(Y, \rho)$  with respect the norm  $\|\cdot\|_{L^p(Y, \rho)}$ .*

*Proof.* Let  $f \in L^p(Y, \rho)$ . For  $k \in \mathbb{N}$  we put

$$f_k(x) = \begin{cases} k & \text{if } f(x) > k \\ f(x) & \text{if } f(x) \in [-k, k] \\ -k & \text{if } f(x) < -k \end{cases}$$

so that  $f_k(x)$  is bounded and measurable. Then by **Lemma 2** there exists  $\tilde{f}_k \in BUC(Y)$  such that

$$\rho(\{x \in Y : \tilde{f}_k(x) \neq f_k(x)\}) \leq \frac{1}{2^k}$$

Then by [12] there exists  $g_k \in Lip_b(Y)$  such that

$$\|g_k - \tilde{f}_k\|_{L^\infty(Y)} \leq \frac{1}{2^k}.$$

Now we estimate

$$\|g_k - f\|_{L^p(Y, \rho)} \leq \|g_k - \tilde{f}_k\|_{L^p(Y, \rho)} + \|\tilde{f}_k - f_k\|_{L^p(Y, \rho)} + \|f_k - f\|_{L^p(Y, \rho)},$$

where

$$\begin{aligned} \|g_k - \tilde{f}_k\|_{L^p(Y, \rho)} &= \left( \int_Y |g_k(x) - \tilde{f}_k(x)|^p \rho(dx) \right)^{1/p} \\ &\leq \|g_k - \tilde{f}_k\|_{L^\infty(Y)} \rho(Y)^{1/p} \leq \frac{\rho(Y)^{1/p}}{2^k}. \end{aligned}$$

Concerning the second term we recall that

$$\sup_{x \in Y} |\tilde{f}_k(x)| \leq 2 \sup_{x \in Y} |f_k(x)| = 2k$$

then

$$\begin{aligned} \|\tilde{f}_k - f_k\|_{L^p(Y, \rho)} &= \left( \int_Y |\tilde{f}_k(x) - f_k(x)|^p \rho(dx) \right)^{1/p} \\ &= \left( \int_{\{x \in Y : \tilde{f}_k(x) \neq f_k(x)\}} |\tilde{f}_k(x) - f_k(x)|^p \rho(dx) \right)^{1/p} \\ &\leq 3k \rho(\{x \in Y : \tilde{f}_k(x) \neq f_k(x)\})^{1/p} \leq \frac{3k}{2^{k/p}}. \end{aligned}$$

Finally we remark that since  $f_k \rightarrow f$   $\rho$ -a.e. for  $k \rightarrow \infty$ , and  $|f_k(x)| \leq |f(x)| \in L^p(Y, \rho)$ , then the Lebesgue theorem yields

$$\|f_k - f\|_{L^p(Y, \rho)} \rightarrow 0, \quad k \rightarrow \infty.$$

□

## References

- [1] V. I. Bogachev, *Gaussian Measures*, American Mathematical Society, USA, 1998.
- [2] G. Cappa *On the Ornstein-Uhlenbeck operator in convex sets of Banach spaces*, pre-print, arXiv:1503.02836.
- [3] P. Celada, A. Lunardi *Traces of Sobolev functions on regular surfaces in infinite dimensions*, J. Funct. Anal. 266 (2014), n. 4, 1948-1987.
- [4] G. Da Prato, A. Lunardi, *Elliptic operators with unbounded drift coefficients and Neumann boundary condition*, J. Differential Equations 198 (2004), 35–52.
- [5] G. Da Prato, A. Lunardi, *Maximal  $L^2$  regularity for Dirichlet problems in Hilbert spaces*, J. Math. Pures Appl. 99 (2013), 741-765.
- [6] G. Da Prato, A. Lunardi, *Maximal Sobolev regularity in Neumann problems for gradient systems in infinite dimensional domains*, Ann. Inst. H. Poincaré Probab. Statist., 51, n. 3 (2015), 1102-1123.



- [7] D. Feyel, A. de La Pradelle, *Hausdorff measures on the Wiener space*, Pot. Analysis 1 (1992), 177-189.
- [8] A. Lunardi, M. Miranda and D. Pallara, *BV functions on convex domain in Wiener spaces*, Potential Analysis, 43 (2015), 23-48.
- [9] M. Mandelkern, *On the uniform continuity of Tietze extensions*, Arch. Math. 55 (1990), 387-388.
- [10] G. Metafune, J. Pruess, A. Rhandi, R. Schnaubelt *The domain of the Ornstein-Uhlenbeck operator on an  $L^p$ -space with invariant measure*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 5 (2002), 471-485
- [11] P.-A. Meyer, *Note sur les processus d'Ornstein-Uhlenbeck*, in: Séminaire de Probabilités, XVI, in: Lecture Notes in Math., vol. 920, Springer-Verlag, Berlin, 1982, 95-133.
- [12] R. Miculescu, *Approximations by Lipschitz functions generated by extensions*, Real Anal. Exchange vol. 28, n. 1 (2002), 33-41.